Fast Calculation of the SPT Bispectrum

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We present a numerical code for fast calculation of the oneloop Standard Perturbation Theory (SPT) Bispectrum and Power Spectrum. It computes the 6350 triangles of BOSS, located between $k = 0.0026\,(\mathrm{h/Mpc})$ and $k = 0.21\,(\mathrm{h/Mpc})$ in **35 seconds** for $z = 0$. Our code has been developed in Fortran 90 and Fortran 77 for ΛCDM, and requires a single transfer function, given by, for example, CAMB or CMBFAST.

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I. INTRODUCTION

As Cosmology moves further into its "precision" era, constraining models becomes not only a matter of precise observation but of fast theoretical *generation*, in the sense that slow explorers of parameter space – normally in the form of random walkers – cannot contribute to the community "Posterior". Such slow random walkers are those hampered by a tedious or computationally intense proposal distribution, which often takes the form of a χ^2 statistic, and for today's perturbative schemes, multi-dimensional integrals over Green's functions, observational window functions and mode-mixing functions. In Standard Cosmological Perturbation theory (SPT), the "Green's function" or propagator – in a loose sense – is the Power Spectrum, $P(k)$, and the mode-mixing functions are $\alpha(k, k_1, k_2)$ and $\beta(k, k_1, k_2)$, derived from the Cold Dark Matter (CDM) fluid equations for the overdensity and velocity fields, $\delta(k)$ and $\theta(k)$. These mode mixing functions are used recursively to define the SPT kernels F_n, G_n. which relate linear solutions in overdensity δ_1 and θ_1 to higher order solutions δ_n, θ_n [1], enabling one to probe deeper into the nonlinear – high k – regime. While the 2-point function and Power spectrum have been calculated to very high degrees of accuracy, with very impressive speed [2], the 3-point function and Bispectrum remain a challenge to Cosmological inference via Monte Carlo methods, because a naive, next-to-leading order calculation of the BOSS Bispectrum (≈ 6000 triangles), even with sophisticated integration tools such as CUBA [3] and an SPT framework codified into Fortran, can take up to 6 hours. And this is *without* convolving with the proper selection effects, discrete k-shell binning, and redshift space distortions (RSD). This is a very slow "walker" indeed. We can only hope to generate rigorous theoretical predictions on the BOSS Bispectrum if the boilerplate calculation – that of the SPT integrals – is substantially expedited.

A. The Goal

We seek to compute, as quickly as possible, the oneloop corrections to the SPT Power Spectrum,

$$\begin{aligned} P(k) &= P_0(k) + P_{13}(k) + P_{22}(k) \\ P_{13}(k) &= 6P_0(k) \int d^3q F_3(\mathbf{k}, \mathbf{q}, -\mathbf{q}) P_0(q) \\ P_{22}(k) &= 2 \int d^3q F_2(\mathbf{q}, \mathbf{k}-\mathbf{q})^2 P_0(q) P_0(|\mathbf{k}-\mathbf{q}|) \end{aligned}$$

and Bispectrum:

$$\begin{aligned} B(\mathbf{k}_1, \mathbf{k}_2) &= B_{112}(\mathbf{k}_1, \mathbf{k}_2) + B_{222}(\mathbf{k}_1, \mathbf{k}_2) + B_{321_I}(\mathbf{k}_1, \mathbf{k}_2) + B_{321_{II}}(\mathbf{k}_1, \mathbf{k}_2) + B_{411}(\mathbf{k}_1, \mathbf{k}_2) \\ B_{112}(\mathbf{k}_1, \mathbf{k}_2) &= 2F_2(\mathbf{k}_1, \mathbf{k}_2) P_0(k_1) P_0(k_2) + 2 \text{ cyc.} \\ B_{222}(\mathbf{k}_1, \mathbf{k}_2) &= 8 \int d^3q F_2\left[\mathbf{q}, -(\mathbf{q}+\mathbf{k}_2)\right] F_2\left[\mathbf{k}_3 - \mathbf{q}, (\mathbf{q}+\mathbf{k}_2)\right] F_2\left[\mathbf{q}-\mathbf{k}_3, \mathbf{q}\right] P_0(q) P_0(|\mathbf{q}-\mathbf{k}_3|) P_0(|\mathbf{q}+\mathbf{k}_2|) \\ B_{321_I}(\mathbf{k}_1, \mathbf{k}_2) &= 6P_0(k_2) \int d^3q F_3\left[\mathbf{k}_3, \mathbf{q}, \mathbf{k_1} - \mathbf{q}\right] F_2\left[\mathbf{k}_1 - \mathbf{q}, \mathbf{q}\right)] P_0(q) P_0(|\mathbf{k}_1 - \mathbf{q}|) + \{\mathbf{k}_1 \leftrightarrow \mathbf{k}_2\} \ + 2 \text{ cyc.} \\ B_{321_{II}}(\mathbf{k}_1, \mathbf{k}_2) &= 6F_2(\mathbf{k}_1, \mathbf{k}_2) P_0(k_1) P_0(k_2) \left(\int d^3q F_3(\mathbf{k_2}, \mathbf{q}, -\mathbf{q}) P_0(q) + \{\mathbf{k}_1 \leftrightarrow \mathbf{k}_2\} \right) + 2 \text{ cyc.} \\ B_{411}(\mathbf{k}_1, \mathbf{k}_2) &= 12P_0(k_1) P_0(k_2) \int d^3q F_4(\mathbf{k_1}, \mathbf{k_2}, \mathbf{q}, -\mathbf{q}) P_0(q) + 2 \text{ cyc.} \end{aligned}$$

There are eighteen Bispectrum diagrams in all: three from the tree level linear Bispectrum, B_{112} – which do not require integration over internal momenta – one from B_{222}, six from B_{321_I}, six from $B_{321_{II}}$ and three from the B_{411}. The game of speeding up our walkers is that of organizing and simplifying these diagrams.

II. TADPOLES: THE P_{13} AND $B_{321_{II}}$

It is now common to identify the diagrams of SPT as reducible or irreducible, based on whether they can be absorbed by the renormalized propagator [4]. In the case of the Zel'dovich approximation, this idea works out quite cleanly, since the Zeldovich kernels have the convenient, separable property:

$$F_{2n+1}(\mathbf{q}_1, -\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_2, \ldots \mathbf{q}_n, -\mathbf{q}_n, \mathbf{k}) = \frac{1}{n!} \prod_{i=1}^{n} \left(\frac{-(\mathbf{k} \cdot \mathbf{q_i})^2}{q_i^4} \right) \tag{1}$$

which, when integrated against linear Power spectra gives a product of σ_v^2's, the variance of the velocity field:

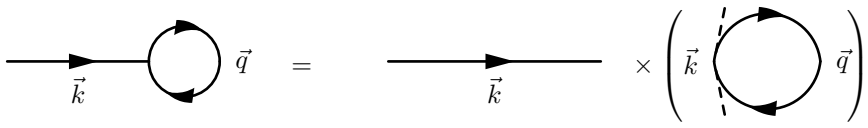
$$\int F_{2n+1}(\mathbf{q}_1, -\mathbf{q}_1 \dots \mathbf{q}_n, -\mathbf{q}_n, \mathbf{k}) P_0(q_1) \cdots P_0(q_n) d^3q_1 \cdots d^3q_n = \frac{1}{n!}(-k^2\sigma_v^2)^n \tag{2}$$

This means that all two-point tadpole diagrams in Zel'dovich – a single internal line with momenta $\mathbf{k}$ connecting any number of closed loops at the ends – are simply products of this $k^2\sigma_v^2$ factor and the linear Power Spectrum. Diagrammatically speaking, this means one can make a "cut" – which we denote by a dashed line – between the linear propagator $P_0(k)$ and the tadpole loops. If you sum all Zel'dovich tadpoles – $P_0 + P_{13} + P_{15} + P_{17} + P_{19} + \dots$ – you get the exponential damping factor from RPT, $P_0(k)e^{-k^2\sigma_v^2}$. In the case of the P_{13} diagram for the Power Spectrum, we have:

$$P_{13}(k) = P_0(k)\left(-k^2\sigma_v^2\right) \qquad \text{ZA} \tag{3}$$

$$= P_0(k) \times \left(6\int d^3q F_3(\mathbf{k}, \mathbf{q}, -\mathbf{q}) P_0(q)\right) \quad \text{SPT} \tag{4}$$

Or, in graphical form,



(5)

Where the integral over internal momenta can be reduced to one dimension:

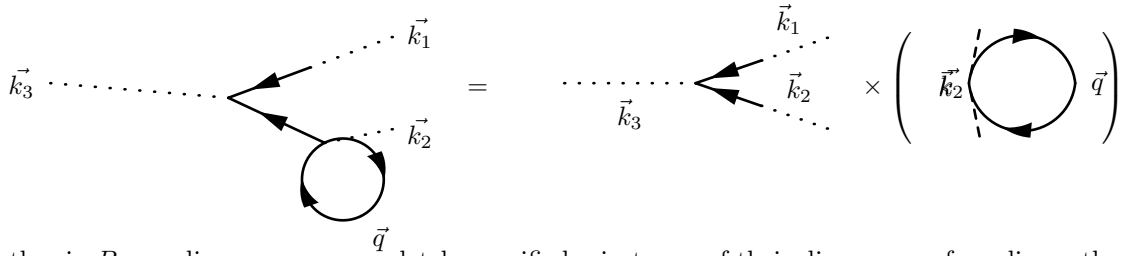
$$P_{13}(k) = 6P_0(k)\int dq P_0(q)\pi \frac{6\left(2k^2+7q^2\right)\left(k^2-q^2\right)^3 \log\left(\frac{|k-q|}{|k+q|}\right) + 4\left(6k^7q - 79k^5q^3 + 50k^3q^5 - 21kq^7\right)}{1512(kq)^3} \tag{6}$$

We take this idea of reducing diagrams to product expressions to the Bispectrum, where the $B_{321_{II}}$ can be built out of linear terms – namely the Power spectrum $P_0(k)$ and F_2 kernel – and a first order correction to the propagator,

$$B_{321_{II}}(\mathbf{k}_1, \mathbf{k}_2) = F_2(\mathbf{k}_1, \mathbf{k}_2)\left(P_0(k_1)P_{13}(k_2) + P_0(k_2)P_{13}(k_1)\right) + 2 \text{ cyc.} \tag{7}$$

$$= B_{112}(\mathbf{k}_1, \mathbf{k}_2)\left(\frac{P_{13}(k_1)}{2P_0(k_1)} + \frac{P_{13}(k_2)}{2P_0(k_2)}\right) \tag{8}$$

Or, to show things diagrammatically, the first term in Equation 7 can be "cut" as the product:



(9)

Thus, the six $B_{321_{II}}$ diagrams are completely specified – in terms of their discrepancy from linear theory – by the P_{13}. In a code that computes the one loop power spectrum *first*, storing P_{13} values in an array, this nullifies the $B_{321_{II}}$ computation time.

III. VERTEX RENORMALIZATION AND B_{411}

For the B_{411} diagram, in the Zel'dovich approximation, one can separate the F_4 kernel,

$$12F_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}, -\mathbf{q}) = 6F_2(\mathbf{k}_1, \mathbf{k}_2)F_3(\mathbf{k}_3, \mathbf{q}, -\mathbf{q}) \tag{10}$$

allowing us to write:

$$B_{411} = B_{112}(\mathbf{k}_1, \mathbf{k}_2)\frac{P_{13}(k_3)}{2P_0(k_3)} + 2 \text{ cyc.} \tag{11}$$

Once again, we have "deleted" a diagram's computation time by reducing it to simpler – and previously calculated – constituents. In the Zel'dovich approximation, the B_{411} is fully specified by the first order correction to the propagator, but in SPT things are not quite so easy. We need to include a vertex correction, which is given by the functional derivative of our fully non-linear overdensity field, taken along with an expectation value:

$$\mathcal{F}_n = \langle \frac{\partial^n \delta\,(\mathbf{k})}{\partial\delta_1(\mathbf{k}_1)\cdots\partial\delta_1(\mathbf{k}_n)} \rangle \tag{12}$$

For the $\mathcal{F}_2$ vertex, our first two terms – to tree level and one loop order – are:

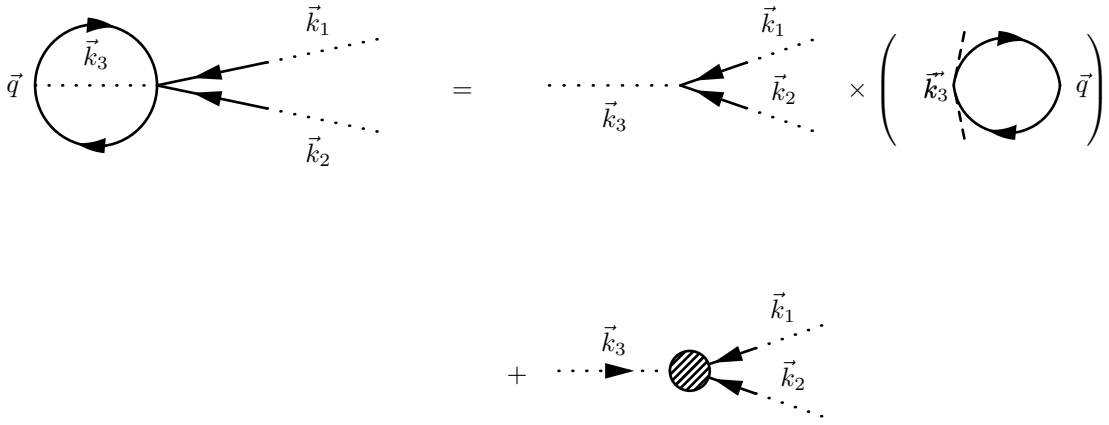
$$\mathcal{F}_2 = F_2(\mathbf{k}_1, \mathbf{k_2}) + \int d^3q\ P_0(q)\,(6F_4((\mathbf{k}_1, \mathbf{k_2}, \mathbf{q}, -\mathbf{q}) - \ 3F_3(\mathbf{k}_3, \mathbf{q}, -\mathbf{q})F_2(\mathbf{k}_1, \mathbf{k}_2)) \tag{13}$$

and this oneloop term is precisely what appears in the B_{411}, if we re-organize things,

$$B_{411} = B_{112}(\mathbf{k}_1, \mathbf{k}_2)\frac{P_{13}(k_3)}{2P_0(k_3)} + 2F_2^{\rm VR}(\mathbf{k}_1, \mathbf{k}_2)P_0(k_1)P_0(k_2) \tag{14}$$

$$F_2^{\rm VR} = \int d^3q\ P_0(q)\ \ (6F_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}, -\mathbf{q}) - 3F_3(\mathbf{k}_3, \mathbf{q}, -\mathbf{q})F_2(\mathbf{k}_1, \mathbf{k}_2)) \tag{15}$$

which can be represented diagrammatically:



(16)

So the B_{411} diagram can be written in terms of oneloop corrections to the propagator, P_{13}, and vertex, $F_2^{\rm VR}$, where the latter goes to zero in the Zel'dovich approximation. The first term in equation (14) can be calculated instantaneously – assuming pre-computed P_{13} values – and the second can be simplified to a one dimensional integral. Once again we have – almost – avoided three-dimensional integration over internal momenta $\mathbf{q}$.

IV. THE P_{22} AND B_{321_I} DIAGRAMS

Now the P_{22} diagram is the simplest example of an "irreducible". In both the Zel'dovich approximation and SPT, we are stuck with what can be reduced to a two-dimensional integral:

$$P_{22}(k) = 2\int d^3q\ F_2(\mathbf{k}-\mathbf{q},\mathbf{q})^2 P_0(q)P_0(|\mathbf{k}-\mathbf{q}|) \tag{17}$$

Defining our second internal momenta as $\mathbf{p} = \mathbf{k} - \mathbf{q}$, we have:

$$P_{22}(k) = \int_{k_{\min}}^{k_{\max}} dq \int_{\max(k_{\min},|k-q|)}^{\min(k_{\max},|k+q|)} dp\ \frac{4\pi pq}{k}\ P_0(p)P_0(q)\hat{F}_2(k,q,p)^2 \tag{18}$$

Where one can define the F_2 kernel in terms of three scalar variables, that of the total incoming – external — momenta and the two outgoing momenta: $F_2(\mathbf{q},\mathbf{p}) = \hat{F}_2(k_{\text{in}}, q_{\text{out}}, p_{\text{out}})$. Since the integrand is symmetric with respect to scalar $p \leftrightarrow q$ exchange, we can put a heaviside $\Theta(p-q)$ in this integrand and multiply by two without consequence:

$$P_{22}(k) = \int_{k_{\min}}^{k_{\max}} dq \int_{\max(q,|k-q|)}^{\min(k_{\max},|k+q|)} dp\ \frac{8\pi pq}{k}\ P_0(p)P_0(q)\hat{F}_2(k,q,p)^2 \tag{19}$$

As for the B_{321_I} diagram, we need six diagrams which are all quite similar to the P_{22}:

$$B_{321_I}(\mathbf{k}_1,\mathbf{k}_2) = 6P_0(k_2)\int d^3q F_3\left[\mathbf{k}_2,\mathbf{q},\mathbf{k_1}-\mathbf{q}\right] F_2\left[\mathbf{k}_1-\mathbf{q},\mathbf{q}\right)] P_0(q)P_0(|\mathbf{k}_1-\mathbf{q}|) + \tag{20}$$

$$6P_0(k_1)\int d^3q F_3\left[\mathbf{k}_1,\mathbf{q},\mathbf{k_2}-\mathbf{q}\right] F_2\left[\mathbf{k}_2-\mathbf{q},\mathbf{q}\right)] P_0(q)P_0(|\mathbf{k}_2-\mathbf{q}|) \tag{21}$$

Let us define $\mathbf{p}_i = \mathbf{k}_i - \mathbf{q}$, and group the six B_{321_I} diagrams into pairs that integrate over same internal momenta $\mathbf{q}, \mathbf{p}_i$:

$$\begin{aligned} B_{321_I} &= 6P_0(k_2)\int d^3q F_3(\mathbf{k}_2,\mathbf{p}_1,\mathbf{q})F_2(\mathbf{p_1},\mathbf{q})P_0(q)P_0(p_1) + \\ &\quad 6P_0(k_3)\int d^3q F_3(\mathbf{k}_3,\mathbf{p}_1,\mathbf{q})F_2(\mathbf{p_1},\mathbf{q})P_0(q)P_0(p_1) + 2\text{ cyc.} \qquad (22) \\ &= 6\int d^3q \left(P_0(k_2)F_3(\mathbf{k}_2,\mathbf{p}_1,\mathbf{q}) + P_0(k_3)F_3(\mathbf{k}_3,\mathbf{p}_1,\mathbf{q})\right) F_2(\mathbf{p_1},\mathbf{q})P_0(q)P_0(p_1) + 2\text{ cyc.} \qquad (23) \end{aligned}$$

Just like the P_{22}, we can reduce each of these integrals – there will be three of them – to two dimensions: that of the scalar variables p_i and q, and also take advantage of $p_i \leftrightarrow q$ symmetry by cutting the integration region in half with a heaviside $\Theta(p_i - q)$.

$$\begin{aligned} B_{321_I} = \frac{24\pi}{k_1}\int_{k_{\min}}^{k_{\max}} qdq \int_{\max(q,|k_1-q|)}^{\min(k_{\max},|k_1+q|)} p_1dp_1 \left[P_0(k_2)\hat{F}_3(k_3,p_1,q,k_2) + P_0(k_3)\hat{F}_3(k_2,p_1,q,k_3)\right] \\ \times\hat{F}_2(k_1,q,p_1)P_0(q)P_0(p_1) + 2\text{ cyc.} \end{aligned} \tag{24}$$

Where, once again we've denoted the angle-integrated F_3 kernel, which is a function of the four scalar momenta variables, as $\hat{F}_3$. One should make sure to never redundantly compute B_{321_I} diagrams. For example, for equilateral triangles $k_1 = k_2 = k_3$, all cyclic permutations of equation 24 are exactly the same, so we need only compute *one* two-dimensional integral. In the case of an isosceles triangle, $k_1 = k_2$, $k_1 = k_3$ or $k_2 = k_3$, *one* of the above cyclic permutations is redundant, and so we need only calculate *two* two-dimensional integrals. Such considerations are important, since the B_{321_I} is so far our most "expensive" diagram.

V. THE B_{222}

The single B_{222} diagram presents the bulk of our oneloop Bispectrum computation time, because it is a three-dimensional integral. In this case, the only simplification that can be made, aside from straightforward integration, is expansion of the integrand from colinearity. In our code we define the internal momenta $\mathbf{q}$ by placing $\mathbf{k}_1$ along the z-axis, and requiring $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$.

$$\begin{aligned}
\mathbf{k_1} &= k_1 \ \langle 0, 0, 1 \rangle \\
\mathbf{k_2} &= k_2 \ \langle \sqrt{1 - x_{12}^2}, 0, x_{12} \ \rangle \\
\mathbf{q} &= q \ \langle \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta \rangle \\
\mathbf{k_1} \cdot \mathbf{k_2} &= k_1 k_2 x_{12} \\
\mathbf{q} \cdot \mathbf{k_1} &= \frac{p^2 - q^2 - k_1^2}{2} = pq\cos\theta
\end{aligned}$$

Sufficient variables to describe our region of integration, for the B_{222} are scalar momenta $q = |\mathbf{q}|$, $p_1 = |\mathbf{q} \pm \mathbf{k}_1|$ and the angle $x_2 = \hat{\mathbf{q}} \cdot \hat{\mathbf{k}_2}$. In the case of a colinear triangle, $|\vec{k}_1| = |\vec{k}_2| + |\vec{k}_3|$, we have $x_{12} = -1$ and so our third variable can be written $x_2 = \hat{\mathbf{q}} \cdot \hat{\mathbf{k}_2} = -\hat{\mathbf{q}} \cdot \hat{\mathbf{k}_1} = -x_1$, which is a function of k_1, q, p_1. Thus, one can reduce the B_{222} integration to two-dimensions for colinear triangles.

For squeezed triangles, $|x_{12}| << 1$, one can write the B_{222} integrand in terms of scalar momenta q, p_1, a single angle ϕ, and expand in x_{12}. Each power of x_{12} will be accompanied by a $\cos\phi$, and so one can integrate analytically in ϕ to reduce numerical integation to two dimensions.

VI. INTEGRATION LIBRARIES

Our code uses the CUBA integration library [3] to compute the standard SPT integrals. We highly suggest CUBA's routine CUHRE for the SPT integrals between two and three dimensions. Otherwise, for single dimensional integrals such as the P_{13}, we use CUBA's VEGAS or dqage.f, the Gauss-Kronrod routine provided by Quadpack. Repeated one-dimensional integrals for the same integrand can be expedited by saving former importance samplings in a VEGAS grid [3].

SUAVE and VEGAS [5] are both Bayesian importance samplers, and therefore sensitive to integrand singularites. For scale free Power spectra $P(k) \sim k^n, n < -1$, infrared divergences – that of low q or when q approaches external momenta k – can make it difficult for VEGAS and SUAVE to converge. In such cases, we suggest "subtracting the singularity" under the integral sign and then adding up all – slightly perturbed – diagrams to get the correct physical result. By virtue of Galilean invariance, the infrared divergences of P_{22} and P_{13} add up to zero; and this holds for all $n-$point functions, at all orders of expansion parameter (which is the growth factor D_+) in SPT. For CDM Power spectra, $P(k) \sim k$ at low k, and so this effect is negligible.

VII. CROSS CHECK AND COMPARISONS WITH N-BODY

ACKNOWLEDGMENTS

We would like to thank Thomas Hahn for enormously helpful comments with the CUBA integration library.

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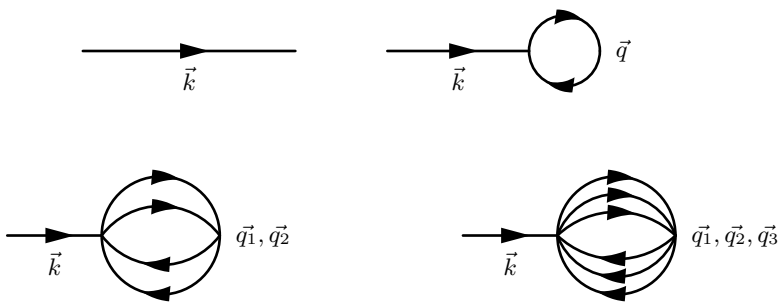

FIG. 1. A few Power Spectrum diagrams: linear power and the first three tadpoles: P_0, P_{13}, P_{15} and P_{17}

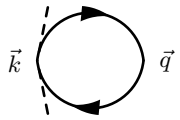

FIG. 2. The reducible part of the P_{13} diagram; a oneloop correction to the two-point propagator, which in Zel'dovich gives the variance of the velocity field, $\int F_3(\mathbf{k}, \mathbf{q}, -\mathbf{q}) P_0(q) d^3q \sim -k^2\sigma_v^2$. The dashed line above represents a diagrammatic "cut". $\vec{k}$ is the incoming external momenta.

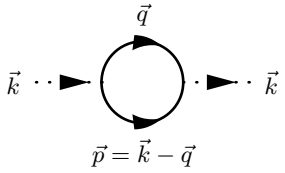

FIG. 3. The irreducible P_{22} diagram, with internal momenta q, p.

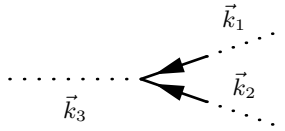

FIG. 4. Tree level Bispectrum diagram, $B_{112} = 2F_2(\mathbf{k}_1, \mathbf{k}_2) P_0(k_1) P_0(k_2) + 2$ cyc..

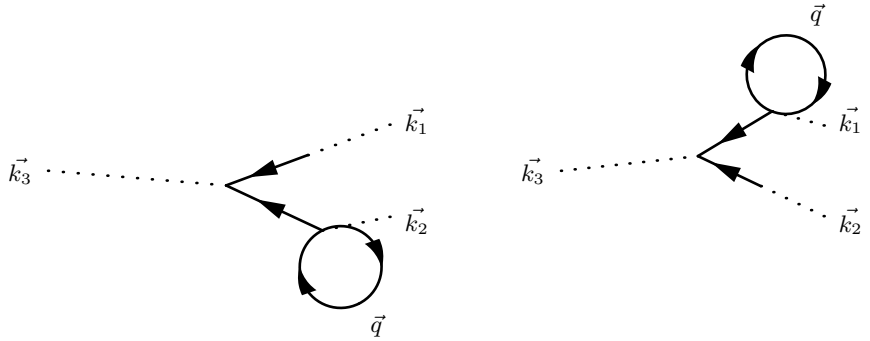

FIG. 5. Two of the six $B_{321_{II}}$ diagrams, which are essentially the tree level diagram times a single tadpole, much like the P_{13}.

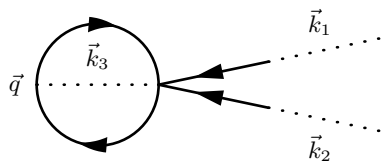

FIG. 6. One of the three B_{411} diagrams. Notice the similarity to the the $B_{321_{II}}$ and P_{13}, except now the loop integration is located at an F_2 vertex.

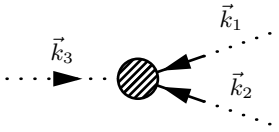

FIG. 7. $F_2^{\rm VR}$, the renormalized vertex.

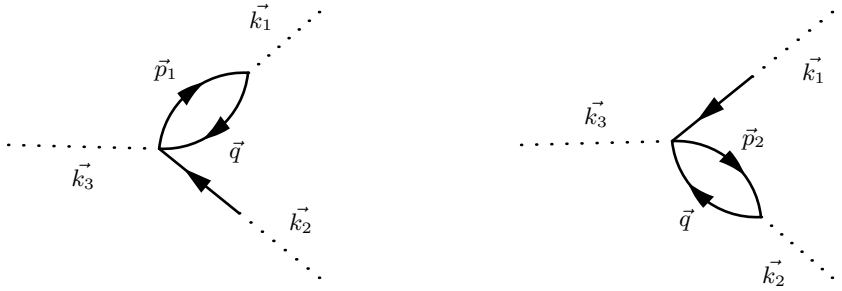

FIG. 8. Two of the six B_{321_I} diagrams, which have similar properties to the irreducible P_{22} two-point diagram. It is very important to group these six diagrams by common loop momenta $\vec{q}$ and $\vec{p}_i = \vec{q} - \vec{k}_i$.

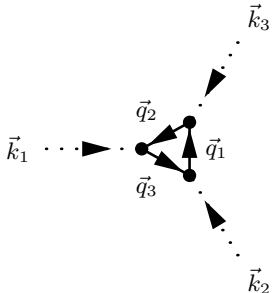

FIG. 9. The irreducible B_{222} diagram, with three ineluctable degrees of freedom to integrate over. The numerical definition of $\{\mathbf{q}_i\}$ turns out to be important for computational speed.